ON ALGEBRAIC EQUATIONS SATISFIED BY HYPERGEOMETRIC SOLUTIONS OF THE QKZ EQUATION.

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October, 1997

Abstract. We consider the $sl(2)$ quantized Knizhnik-Zamolodchikov equation (qKZ), defined in terms of rational R-matrices. The properties of the equation change when the step of the equation takes a resonance value. In this case the discrete connection defined by the qKZ equation has a invariant subbundle which we call the subbundle of quantized conformal blocks. Solutions of the qKZ equation were constructed in [TV1],[MV1] in terms of multidimensional hypergeometric integrals. In this paper we show that for a resonance step all hypergeometric solutions take values in the subbundle of quantized conformal blocks, moreover the values span the subbundle of quantized conformal blocks under certain conditions. We describe the space of hypergeometric solutions in terms of the quantum group $U_q sl(2)$.

1. Introduction

Conformal field theory associates a finite dimensional vector space, called the space of conformal blocks, to each Riemann surface with marked points and certain additional data (local coordinates, representations). The vector spaces corresponding to different complex structures or different positions of the points are locally (projectively) identified by a projectively flat connection.

A Wess-Zumino-Witten model is labeled by a simple Lie algebra $\mathfrak{g}$ and a positive integer c called level. The space of conformal blocks is defined in terms of the representation theory of the affine Kac-Moody algebra $\widehat{L\mathfrak{g}}$, which is a central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$, see [K]. For each irreducible highest weight $\mathfrak{g}$-module L, we have a canonically defined corresponding irreducible highest weight $\widehat{L\mathfrak{g}}$ module of level c, denoted $\widehat{L}$, see [K]. Then, the space of conformal blocks associated to a Riemann surface Σ, n distinct points with a choice of local holomorphic parameters around them and n irreducible finite dimensional $\mathfrak{g}$ modules $L_1, \dots, L_n$ (such that $\widehat{L}_i$ are integrable) is the space $H^0(\mathfrak{g}(\Sigma), (\widehat{L}_1 \otimes \dots \otimes \widehat{L}_n)^*)$ of linear forms on $\widehat{L}_1 \otimes \dots \otimes \widehat{L}_n$ invariant under the action of the Lie algebra $\mathfrak{g}(\Sigma)$ of meromorphic $\mathfrak{g}$-valued functions on Σ, holomorphic outside of the marked points. The action of $\mathfrak{g}(\Sigma)$ is defined through the Laurent expansion at the poles. Varying the data makes the spaces of conformal blocks into a holomorphic vector bundle with a projectively flat connection given by the Sugawara-Segal construction.

An explicit description is known on the Riemann sphere $\mathbb{P}^1$. Namely, the space of conformal blocks on $\mathbb{P}^1$ with $n+1$ distinct points $z_1, \dots, z_n \in \mathbb{C} \subset \mathbb{P}^1$, $z_{n+1} = \infty$,

associated to $n+1$ irreducible $\mathfrak{g}$ modules $L_1, \dots, L_{n+1}$ with highest weights $\lambda_1, \dots, \lambda_{n+1}$, is identified with a subspace of $(L_1 \otimes \dots \otimes L_n)^{sing}_\lambda$, where $(L_1 \otimes \dots \otimes L_n)^{sing}_\lambda \subset L_1 \otimes \dots \otimes L_n$ is the weight subspace of singular vectors of total weight λ, $\lambda = -w\lambda_{n+1}$ and w is the longest element of the Weyl group. More precisely the space of conformal blocks can be described as follows. Let θ be the highest root and let the scalar product be normalized by $(\theta, \theta) = 2$.

If the resonance condition,

$$(\theta, \lambda) - c + k - 1 = 0, \tag{1}$$

holds for some $k \in \mathbb{N}$, then the space of conformal blocks $N_{\lambda_{n+1}}(z)$ is identified with the subspace

$$N_{\lambda_{n+1}}(z) = \{m \in (L_1 \otimes \dots \otimes L_n)^{sing}_\lambda \mid (E(z))^k m = 0\},$$

otherwise the space of conformal blocks is identified with the weight space of singular vectors, $N_{\lambda_{n+1}}(z) = (L_1 \otimes \dots \otimes L_n)^{sing}_\lambda$. Here

$$E(z) = \sum_{i=0}^{n} z_i e_\theta^{(j)},$$

and $e_\theta^{(j)}$ denotes the element $e_\theta \in \mathfrak{g}$ acting on the j-th factor, see [KT], [FSV1].

The subspaces $N_{\lambda_{n+1}}(z)$ for different sets of $z_1, \dots, z_n \in \mathbb{C}$ form a subbundle of the trivial vector bundle over the space $\mathbb{C}^n$ with fiber $L_1 \otimes \dots \otimes L_n$. There is a flat connection on the bundle $\mathbb{C}^n \times (L_1 \otimes \dots \otimes L_n)$ preserving the subbundle of conformal blocks. Its horizontal sections $\Psi(z)$ obey the Knizhnik-Zamolodchikov equation,

$$\partial_{z_i} \Psi(z) = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Psi(z), \qquad i = 1, \dots, n, \tag{2}$$

$\kappa = c + h^\vee$, where Ω_{ij} is the Casimir operator acting on the i-th and j-th factors and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

In [SV] the KZ equation was solved in terms of hypergeometric integrals, cf. [R]. It was shown in [V] that for generic values of κ, the space of hypergeometric solutions can be naturally identified with the space of singular vectors in the tensor product $L_1^q \otimes \dots \otimes L_n^q$ of the corresponding representations of the quantum group $U_q\mathfrak{g}$ with $q = e^{-\pi i/\kappa}$.

In [FSV1-3] it was shown that under resonance condition (1), all hypergeometric solutions of the KZ equation with values in $(L_1 \otimes \dots \otimes L_n)^{sing}_\lambda$ automatically take values in the space of conformal blocks, i.e. their values lie in the kernel of the operator $(E(z))^k$. Moreover, in [V] it was shown that for $\mathfrak{g} = sl(2)$, the hypergeometric solutions span the space of conformal blocks and the space of solutions is naturally identified with the co-image of the operator $(f_q)^k \in U_q sl(2)$,

$$(L_1^q \otimes \dots \otimes L_n^q)^{sing}_\lambda / (f_q)^k ((L_1^q \otimes \dots \otimes L_n^q)^{sing}_{\lambda + k\theta}).$$

Let $\mathfrak{g} = sl(2)$. The quantized Knizhnik-Zamolodchikov (qKZ) equation is a holonomic system of difference equations for a function $\Psi(z)$ with values in $L_1 \otimes \dots \otimes L_n$,

$$\Psi(z_1, \dots, z_i + p, \dots, z_n) = R_{m,m-1}(z_m - z_{m-1} + p) \dots R_{m,1}(z_m - z_1 + p) \times$$
$$\times R_{m,n}(z_m - z_n) \dots R_{m,m+1}(z_m - z_{m+1}) \Psi(z),$$

$i = 1, \dots, n$, where $R_{i,j}(x)$ is the rational R-matrix $R_{L_i L_j}(x)$ acting in i-th and j-th factors and $p \in \mathbb{C}$ is a parameter, see [FR].

The qKZ equation defines the discrete Knizhnik-Zamolodchikov (qKZ) connection on the trivial vector bundle over $\mathbb{C}^n$ with fiber $L_1 \otimes \ldots \otimes L_n$.

Consider the quasiclassical limit of the qKZ. Namely, set $y_i = z_i/h$, for some $h \in \mathbb{C}$ and let $h \to 0$. In this limit the qKZ equation turns into a system of differential equations

$$p\partial_{y_i}\tilde{\Psi}(y) = -\sum_{j\neq i} \frac{\tilde{\Omega}_{ij}}{y_i - y_j}\tilde{\Psi}(y), \qquad i = 1, \ldots, n, \tag{3}$$

where $\tilde{\Omega}_{ij} = \Omega_{ij} - 2\lambda_i\lambda_j$ see Section 7 in [TV1] and Section 12.5 in [CP].

Notice that if $p = -\kappa$, then $\Psi(z)$ is a solution of the KZ equation (2) if and only if the function

$$\tilde{\Psi}(y) = \prod_{i<j}(y_i - y_j)^{-2\lambda_i\lambda_j/\kappa}\,\Psi(y)$$

is a solution of the equation (3).

In [TV1], [MV1] the qKZ equation was solved in terms of hypergeometric integrals. It was shown that the space of hypergeometric solutions for generic values of p can be naturally identified with the space of singular vectors in the tensor product $L_1^q \otimes \ldots \otimes L_n^q$ of the corresponding representations of the quantum group $U_q sl(2)$ with $q = e^{\pi i/p}$.

In [MV2] a quantization of the space of conformal blocks is suggested. Namely, under the resonance condition,

$$2\lambda + p + N + k - 1 = 0, \tag{4}$$

where $k \in \mathbb{N}$, we define the space of quantized conformal blocks, $C_{\lambda_{n+1}}(z)$, as

$$C_{\lambda_{n+1}}(z) = \{m \in (L_1 \otimes \ldots \otimes L_n)_\lambda^{sing} \,|\, (e(z))^k m = 0\},$$

where

$$e(z)m = \sum_{j=1}^{n}\left(z_j + h^{(j)} + \sum_{s=j+1}^{n} 2h^{(s)}\right)e^{(j)}m,$$

and $h^{(s)}, e^{(s)}$ denote the elements $h, e \in sl(2)$ acting on the s-th factor of $L_1 \otimes \ldots \otimes L_n$.

The operator $e(z)$ can be described in terms of the action of the Yangian $Y(gl(2))$ in the tensor product of evaluation modules $L_1(z_1)\otimes\ldots\otimes L_n(z_n)$, $e(z) = T_{21}^{(2)} - T_{22}^{(1)}T_{21}^{(1)}$, see (18). In the quasiclassical limit the resonance condition (4) coincides with the resonance condition (1) and the operator $e(z)$ tends to the operator $E(z)$.

It was shown in [MV2] that the spaces of quantized conformal blocks form a subbundle invariant with respect to the quantized KZ connection.

In this paper we show that under resonance condition (4), all hypergeometric solutions of the $sl(2)$ qKZ equation with values in $(L_1 \otimes \ldots \otimes L_n)_\lambda^{sing}$ automatically take values in the space of quantized conformal blocks. We identify the space of the hypergeometric solutions with the co-image of the operator $(f_q)^k \in U_q sl(2)$,

$$(L_1^q \otimes \ldots \otimes L_n^q)_\lambda^{sing}/(f_\theta^q)^k((L_1^q \otimes \ldots \otimes L_n^q)_{\lambda+k\theta}^{sing}).$$

We prove that under certain conditions the hypergeometric solutions span the space of quantized conformal blocks.

The paper is organized as follows. In Section 2 we recall some general facts about $sl(2)$, $U_q sl(2)$ and $Y(gl(2))$. In Section 3 we recall the construction of the hypergeometric

solutions of the qKZ equation. In Section 4 we state and prove the main results of this paper.

2. General definitions and notations

2.1. The Lie algebra $sl(2)$. Let e, f, h be generators of the Lie algebra $sl(2)$ such that

$$[h,e]=e, \qquad [h,f]=-f, \qquad [e,f]=2h.$$

For an $sl(2)$ module M, let M^* be its restricted dual with an $sl(2)$ module structure defined by

$$\langle e\varphi,x\rangle=\langle\varphi,fx\rangle, \qquad \langle f\varphi,x\rangle=\langle\varphi,ex\rangle, \qquad \langle h\varphi,x\rangle=\langle\varphi,hx\rangle$$

for all $x\in M$, $\varphi\in M^*$. The module M^* is called the *dual* module.

For $\lambda\in\mathbb{C}$, denote V_λ the $sl(2)$ Verma module with highest weight λ, $V_\lambda=\bigoplus\limits_{i=0}^{\infty}\mathbb{C}f^iv$, where v is a highest weight vector. Denote L_λ the irreducible module with highest weight λ.

Let $\Lambda^+=\{0,\frac12,1,\frac32,2,...\}$ be the set of dominant weights. If $\lambda\in\Lambda^+$, then L_λ is a $(2\lambda+1)$-dimensional module and

$$L_\lambda\simeq V_\lambda/S_\lambda,$$

where $S_\lambda=\bigoplus\limits_{i=2\lambda+1}^{\infty}\mathbb{C}f^iv\subset V_\lambda$ is the maximal proper submodule. The vectors f^iv, $i=0,\dots,2\lambda$, generate a basis in L_λ.

For $\lambda\notin\Lambda^+$, $L_\lambda=V_\lambda$. It is convenient to introduce S_λ to be the zero submodule of V_λ, then $L_\lambda\simeq V_\lambda/S_\lambda$ for all $\lambda\in\mathbb{C}$.

For an $sl(2)$ module M with highest weight λ, denote by $(M)_l$ the subspace of weight $\lambda-l$, by $(M)^{sing}$ the kernel of the operator e, and by $(M)_l^{sing}$ the subspace $(M)_l\bigcap(M)^{sing}$.

2.2. The algebra $U_qsl(2)$. Let q be a complex number different from ±1. Let e_q, f_q, q^h, q^{-h} be generators of $U_qsl(2)$ such that

$$q^hq^{-h}=q^{-h}q^h=1, \qquad [e_q,f_q]=\frac{q^{2h}-q^{-2h}}{q-q^{-1}},$$

$$q^he_q=qe_qq^h, \qquad q^hf_q=q^{-1}f_qq^h.$$

A comultiplication $\Delta: U_qsl(2)\to U_qsl(2)\otimes U_qsl(2)$ is given by

$$\Delta(q^h)=q^h\otimes q^h, \qquad \Delta(q^{-h})=q^{-h}\otimes q^{-h},$$

$$\Delta(e_q)=e_q\otimes q^h+q^{-h}\otimes e_q, \qquad \Delta(f_q)=f_q\otimes q^h+q^{-h}\otimes f_q.$$

The comultiplication defines a module structure on tensor products of $U_qsl(2)$ modules.

For $\lambda\in\mathbb{C}$, denote V_λ^q the $U_qsl(2)$ Verma module with highest weight q^λ, $V_\lambda^q=\bigoplus\limits_{i=0}^{\infty}\mathbb{C}f_q^iv^q$, where v^q is a highest weight vector.

For $\lambda\in\Lambda^+$, $S_\lambda^q=\bigoplus\limits_{i=2\lambda+1}^{\infty}\mathbb{C}f_q^iv^q$ is a submodule in V_λ^q. Denote L_λ^q the quotient module V_λ^q/S_λ^q. The module L_λ^q is the $(2\lambda+1)$-dimensional highest weight module with highest weight q^λ. The vectors $f_q^iv^q$, $i=0,\dots,2\lambda$, generate a basis in L_λ^q.

For $\lambda \notin \Lambda^+$, let $L^q_\lambda = V^q_\lambda$. It is convenient to introduce S^q_λ to be the zero submodule of V^q_λ, then $L^q_\lambda \simeq V^q_\lambda / S^q_\lambda$ for all $\lambda \in \mathbb{C}$.

For an $U_q sl(2)$ module M^q with highest weight q^λ, denote by $(M^q)_l$ the subspace of weight $q^{\lambda-l}$, by $(M^q)^{sing}$ the kernel of the operator e_q, and by $(M^q)^{sing}_l$ the subspace $(M^q)_l \bigcap (M^q)^{sing}$.

2.3. **The Hopf algebra $Y(gl(2))$.** The Yangian $Y(gl(2))$ is an associative algebra with an infinite set of generators $T^{(s)}_{i,j}$, $i,j = 1,2$, $s = 0,1,\dots$, subject to the following relations:

$$[T^{(r)}_{ij}, T^{(s+1)}_{kl}] - [T^{(r+1)}_{ij}, T^{(s)}_{kl}] = T^{(r)}_{kj} T^{(s)}_{il} - T^{(s)}_{kj} T^{(r)}_{il}, \qquad T^{(0)}_{ij} = \delta_{ij},$$

$i,j,k,l = 1,2$; $r,s, = 1,2,\dots$.

The comultiplication $\Delta : Y(gl(2)) \to Y(gl(2)) \otimes Y(gl(2))$ is given by

$$\Delta \,:\, T^{(s)}_{ij} \mapsto \sum_{k=1}^{2} \sum_{r=0}^{s} T^{(r)}_{ik} \otimes T^{(s-r)}_{kj}.$$

For each $x \in \mathbb{C}$, there is an automorpfism $\rho_x : Y(gl(2)) \to Y(gl(2))$ given by

$$\rho_x \,:\, T^{(s)}_{ij} \mapsto \sum_{r=1}^{s} \binom{s-1}{r-1} x^{s-r} T^{(r)}_{ij}.$$

There is also an *evaluation morphism* ϵ to the universal enveloping algebra of $sl(2)$, $\epsilon : Y(gl(2)) \to U(sl(2))$, given by

$$\epsilon \,:\, T^{(s)}_{11} \mapsto \delta_{1s} h, \qquad \epsilon \,:\, T^{(s)}_{12} \mapsto \delta_{1s} f,$$

$$\epsilon \,:\, T^{(s)}_{21} \mapsto \delta_{1s} e, \qquad \epsilon \,:\, T^{(s)}_{22} \mapsto -\delta_{1s} h,$$

for $s = 1,2,\dots$.

Introduce the generating series $T_{ij}(u) = \sum_{s=0}^{\infty} T^{(s)}_{ij} u^{-s}$. In terms of these series the relations in the Yangian take the form

$$R(x-y) T_{(1)}(x) T_{(2)}(y) = T_{(2)}(y) T_{(1)}(x) R(x-y),$$

where $R(x) = (x \,\mathrm{Id} + P) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, $P \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is the operator of permutation of the two factors, $T_{(1)}(x) = 1 \otimes T(x)$, $T_{(2)}(x) = T(x) \otimes 1$.

In terms of the generating series the comultiplication Δ, the automorphisms ρ_x and the evaluation morphism ϵ take the form

$$\Delta \,:\, T_{ij}(u) \mapsto \sum_{k=1}^{2} T_{ik}(u) \otimes T_{kj}(u),$$

$$\rho_x \,:\, T_{ij}(u) \mapsto T_{ij}(u-x),$$

$$\epsilon \,:\, T_{11}(u) \mapsto \frac{h}{u}, \qquad \epsilon \,:\, T_{12}(u) \mapsto \frac{f}{u},$$

$$\epsilon \,:\, T_{21}(u) \mapsto \frac{e}{u}, \qquad \epsilon \,:\, T_{22}(u) \mapsto -\frac{h}{u},$$

$i,j = 1,2$. For more detail on the Yangian see [CP],[KR].

For any $sl(2)$ module M and $x \in \mathbb{C}$, let $M(x)$ be the $Y(gl(2))$ module obtained from M via the homomorphism $\epsilon \circ \rho(x)$. The module $M(x)$ is called the *evaluation module*. The action of $Y(gl(2))$ in the evaluation module $M(x)$ is given by

$$T_{11}^{(s)} m = x^{s-1} h\, m, \qquad T_{12}^{(s)} m = x^{s-1} f\, m,$$
$$T_{21}^{(s)} m = x^{s-1} e\, m, \qquad T_{22}^{(s)} m = -x^{s-1} h\, m,$$

for all $m \in M$, $s = 1, 2, \dots$.

For $\lambda_1, \lambda_2 \in \mathbb{C}$ and generic complex numbers x, y, the $Y(gl(2))$ modules $L_{\lambda_1}(x) \otimes L_{\lambda_2}(y)$ and $L_{\lambda_2}(y) \otimes L_{\lambda_1}(x)$ are irreducible and isomorphic. There is a unique intertwiner of the form $PR_{L_{\lambda_1}L_{\lambda_2}}(x-y)$ mapping $v_1 \otimes v_2$ to $v_2 \otimes v_1$, where P is the operator of permutation of the two factors and v_i are highest weight vectors generating L_{λ_i}, $i = 1, 2$. The operator $R_{L_{\lambda_1}L_{\lambda_2}}(x) \in \mathrm{End}(L_{\lambda_1} \otimes L_{\lambda_2})$ is called the *rational R-matrix*, see [CP], [D].

The vector spaces $V_{\lambda_1} \otimes V_{\lambda_2}$ for different values of $\lambda_1, \lambda_2 \in \mathbb{C}$ are identified by distinguished bases $\{f^{l_1} v_1 \otimes f^{l_2} v_2 \,|\, l_1, l_2 \in \mathbb{Z}_{\geq 0}\}$.

Theorem 1. *(Theorem 1 in* [MV1]*.)*

1. The rational R-matrix $R_{V_{\lambda_1}V_{\lambda_2}}(x) \in \mathrm{End}(V \otimes V)$ is a meromorphic function of x, λ_1, λ_2. Moreover, for any $\lambda_1, \lambda_2 \in \mathbb{C}$, the R-matrix $R_{V_{\lambda_1}V_{\lambda_2}}(x)$ is a well defined meromorphic function of x.

2. For generic $x \in \mathbb{C}$, the rational R-matrix $R_{V_{\lambda_1}V_{\lambda_2}}(x)$ preserves the submodule $V_{\lambda_1} \otimes S_{\lambda_2} + S_{\lambda_1} \otimes V_{\lambda_2} \subset V_{\lambda_1} \otimes V_{\lambda_2}$.

3. Let $V_{\lambda_1} \otimes V_{\lambda_2} \to L_{\lambda_1} \otimes L_{\lambda_2}$ be the canonical factorization map. Then, for generic x, the rational R-matrix $R_{V_{\lambda_1}V_{\lambda_2}}(x)$ can be factorized to an operator $R(x) \in \mathrm{End}(L_{\lambda_1} \otimes L_{\lambda_2})$ and, moreover, $R(x) = R_{L_{\lambda_1}L_{\lambda_2}}(x)$.

Let either $M_i = V_{\lambda_i}$, $i = 1, 2$, be Verma $sl(2)$ modules or $M_i = L_{\lambda_i}$, $i = 1, 2$, be irreducible $sl(2)$ modules. For all $x \in \mathbb{C}$, the rational R-matrix $R_{M_1M_2}(x)$ commutes with the action of $sl(2)$ in $M_1 \otimes M_2$ and, in particular, preserves the weight decomposition.

2.4. **The qKZ connection.** The *rational quantized Knizhnik-Zamolodchikov equation (qKZ)* associated to $sl(2)$ is the following holonomic system of linear difference equations for a function $\Psi(z_1, \dots, z_n)$ with values in a tensor product $M_1 \otimes \dots \otimes M_n$ of $sl(2)$ modules:

$$\Psi(z_1, \dots, z_m + p, \dots, z_n) = K_m(z) \Psi(z_1, \dots, z_n), \qquad m = 1, \dots, n,$$

$$K_m(z) = R_{M_m M_{m-1}}(z_m - z_{m-1} + p) \dots R_{M_m M_1}(z_m - z_1 + p) \times$$
$$\times R_{M_m M_n}(z_m - z_n) \dots R_{M_m M_{m+1}}(z_m - z_{m+1}),$$

where p is a complex parameter, $R_{M_iM_j}(x) \in \mathrm{End}(M_i \otimes M_j)$ is the rational R-matrix acting in the i-th and j-th factors, see [FR]. The linear operators $K_i(z)$ are called the *qKZ operators*.

The qKZ operators commute with the $sl(2)$ action in the tensor product $M_1 \otimes \dots \otimes M_n$ and, in particular, preserve the subspaces $(M_1 \otimes \dots \otimes M_n)_l$ and $(M_1 \otimes \dots \otimes M_n)_l^{sing}$ for all $l \in \mathbb{Z}_{\geq 0}$. In order to construct all solutions of the qKZ equation, it is enough to solve the equation with values in singular weight spaces $(M_1 \otimes \dots \otimes M_n)_l^{sing}$.

The qKZ operators define a discrete flat connection on the trivial vector bundle over $\mathbb{C}^n$ with fiber $M_1 \otimes \dots \otimes M_n$. This connection is called the *quantized Knizhnik-Zamolodchikov connection.*

3. Hypergeometric solutions of the qKZ equation.

3.1. **The phase function.** Let $z=(z_1,\dots,z_n)\in\mathbb{C}^n$, $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$, $t=(t_1,\dots,t_l)\in\mathbb{C}^l$. The *phase function* is defined by the following formula:

$$\Phi(t,z,\lambda)=\prod_{i=1}^{n}\prod_{j=1}^{l}\frac{\Gamma((t_j-z_i+\lambda_i)/p)}{\Gamma((t_j-z_i-\lambda_i)/p)}\prod_{1\le i<j\le l}\frac{\Gamma((t_i-t_j-1)/p)}{\Gamma((t_i-t_j+1)/p)}.$$

3.2. **Actions of the symmetric group.** Let $f=f(t_1,\dots,t_l)$ be a function. For a permutation $\sigma\in\mathbb{S}^l$, define the functions $[f]^{rat}_\sigma$ and $[f]^{trig}_\sigma$ via the action of the simple transpositions $(i,i+1)\in\mathbb{S}^l$, $i=1,\dots,l-1$, given by

$$[f]^{rat}_{(i,i+1)}(t_1,\dots,t_l)=f(t_1,\dots,t_{i+1},t_i,\dots,t_l)\frac{t_i-t_{i+1}-1}{t_i-t_{i+1}+1},$$

$$[f]^{trig}_{(i,i+1)}(t_1,\dots,t_l)=f(t_1,\dots,t_{i+1},t_i,\dots,t_l)\frac{\sin(\pi(t_i-t_{i+1}-1)/p)}{\sin(\pi(t_i-t_{i+1}+1)/p)}.$$

If for all $\sigma\in\mathbb{S}^l$, $[f]^{rat}_\sigma=f$, then we say that the function is *symmetric with respect to the rational action*. If for all $\sigma\in\mathbb{S}^l$, $[f]^{trig}_\sigma=f$, then we say that the function is *symmetric with respect to the trigonometric action*.

This definition implies the following important Remark.

Remark. If $w(t_1,\dots,t_l)$ is $\mathbb{S}^l$ symmetric with respect to the rational action and $W(t_1,\dots,t_l)$ is $\mathbb{S}^l$ symmetric with respect to the trigonometric action, then ΦwW is a symmetric function of $t_1,\dots,t_l$ (in the usual sense).

3.3. **Rational weight functions.** Fix natural numbers n,l.

Set $\mathcal{Z}^n_l=\{\bar l=(l_1,\dots,l_n)\in\mathbb{Z}^n_{\ge0}\,|\,\sum_{i=1}^{n}l_i=l\}$. For $\bar l\in\mathcal{Z}^n_l$ and $m=0,1,\dots,n$, set $l^m=\sum_{i=1}^{m}l_i$.

For $\bar l\in\mathcal{Z}^n_l$, define the *rational weight function* $w_{\bar l}$ by $w_{\bar l}=\sum_{\sigma\in\mathbb{S}^l}[\eta_{\bar l}]^{rat}_\sigma$, where

$$\eta_{\bar l}(t,z,\lambda)=\prod_{m=1}^{n}\frac{1}{l_m!}\prod_{j=l^{m-1}+1}^{l^m}\left(\frac{1}{t_j-z_m-\lambda_m}\prod_{k=1}^{m}\frac{t_j-z_k+\lambda_k}{t_j-z_k-\lambda_k}\right).\tag{5}$$

Example. Let $l=1$. Then the rational weight functions $w_{(0,\dots,1_i,\dots,0)}=w_i$ have the form

$$w_i(t,z,\lambda)=\frac{1}{t-z_i-\lambda_i}\prod_{m=1}^{i-1}\frac{t-z_m+\lambda_m}{t-z_m-\lambda_m},\tag{6}$$

$i=1,\dots,n$.

For fixed $z,\lambda\in\mathbb{C}^n$, the space spanned over $\mathbb{C}$ by all rational weight functions $w_{\bar l}(t,z,\lambda)$, $\bar l\in\mathcal{Z}^n_l$, is called the *hypergeometric rational space specialized at* z,λ and is denoted $\mathfrak{F}(z,\lambda)=\mathfrak{F}^n_l(z,\lambda)$. This space is a space of functions of variable t.

3.4. **Trigonometric weight functions.** Fix natural numbers n, l.

For $\bar l \in \mathcal{Z}_l^n$, define the *trigonometric weight function* $W_{\bar l}$ by

$$W_{\bar l}(t,z,\lambda) =$$
$$\sum_{\sigma\in\mathbb{S}^l}\left[\prod_{m=1}^{n}\prod_{s=1}^{l_m}\frac{\sin(\pi/p)}{\sin(\pi s/p)}\prod_{j=l^{m-1}+1}^{l^m}\frac{\exp(\pi i(z_m-t_j)/p)}{\sin(\pi(t_j-z_m-\lambda_m)/p)}\prod_{k=1}^{m}\frac{\sin(\pi(t_j-z_k+\lambda_k)/p)}{\sin(\pi(t_j-z_k-\lambda_k)/p)}\right]_\sigma^{trig}.$$

A function $W(t,z,\lambda)$ is said to be a *holomorphic trigonometric weight function* if

$$W(t,z,\lambda)=\sum_{m_1+\ldots+m_n=l} a_{\bar m}(\lambda, e^{2\pi i z_1/p},\ldots,e^{2\pi i z_n/p})W_{\bar m}(t,z,\lambda), \tag{7}$$

where $a_{\bar m}(\lambda,u)$ are holomorphic functions of parameters $\lambda, u\in\mathbb{C}^n$. We denote $\mathfrak{G}$ the space of all holomorphic trigonometric weight functions. This space is a space of functions of variables t,z,λ.

For fixed $\lambda, z\in\mathbb{C}^n$, the space spanned over $\mathbb{C}$ by all trigonometric weight functions $W_{\bar l}(t,z,\lambda)$, $\bar l\in\mathcal{Z}_l^n$, is called the *hypergeometric trigonometric space specialized at* z,λ and is denoted $\mathfrak{G}(z,\lambda)=\mathfrak{G}_l^n(z,\lambda)$. This space is a space of functions of variable t.

For $\bar l\in\mathcal{Z}_l^{n-1}$, define the *singular trigonometric weight function* $W_{\bar l}^{sing}$ by

$$W_{\bar l}^{sing}(t,z,\lambda)=$$
$$\sum_{\sigma\in\mathbb{S}^l}\left[\prod_{m=1}^{n-1}\prod_{s=1}^{l_m}\frac{\sin(\pi/p)}{\sin(\pi s/p}\sin(\pi(z_m-\lambda_m-z_{m+1}-\lambda_{m+1}+s-1)/p)\times\right.$$
$$\times\prod_{j=l^{m-1}+1}^{l^m}\frac{1}{\sin(\pi(t_j-z_m-\lambda_m)/p)\sin(\pi(t_j-z_{m+1}-\lambda_{m+1})/p)}\times$$
$$\left.\times\prod_{k=1}^{m}\frac{\sin(\pi(t_j-z_k+\lambda_k)/p)}{\sin(\pi(t_j-z_k-\lambda_k)/p)}\right]_\sigma^{trig}.$$

For fixed $z,\lambda\in\mathbb{C}^n$, the space spanned over $\mathbb{C}$ by all singular trigonometric weight functions $W_{\bar l}(t,z,\lambda)$, $\bar l\in\mathcal{Z}_l^n$, is called the *singular hypergeometric trigonometric space specialized at* z,λ and is denoted $\mathfrak{G}^{sing}(z,\lambda)=\mathfrak{G}_l^{sing,n}(z,\lambda)$.

We have $\mathfrak{G}^{sing}(z,\lambda)\subset\mathfrak{G}(z,\lambda)$, see Lemma 2.29 in [TV1].

A function $W(t,z,\lambda)\in\mathfrak{G}$ is said to be a *holomorphic singular trigonometric weight function* if $W(t,z,\lambda)$ is a holomorphic trigonometric weight function, and for all $z,\lambda\in\mathbb{C}^n$, the function $W(t,z,\lambda)$ belongs to $\mathfrak{G}^{sing}(z,\lambda)$. We denote $\mathfrak{G}^{sing}$ the space of all holomorphic singular trigonometric weight functions. This space is a space of functions of variables t,z,λ.

For any $\bar l\in\mathcal{Z}_l^{n-1}$, the function $W_{\bar l}^{sing}(t,z,\lambda)$ belongs to $\mathfrak{G}^{sing}$, see Lemma 4 in [MV1].

Let $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$, $\bar l=(l_1,\ldots,l_n)\in\mathbb{Z}_{\geq 0}^n$. An i-th coordinate of $\bar l$ is called λ-*admissible* if either $\lambda_i\notin\Lambda^+$ or $\lambda_i\in\Lambda^+$ and $l_i\leq 2\lambda_i$. An index $\bar l$ is called λ-*admissible* if all its coordinates are λ-admissible.

For fixed $z,\lambda\in\mathbb{C}^n$, the space spanned over $\mathbb{C}$ by trigonometric weight functions $W_{\bar l}(t,z,\lambda)$ with λ-admissible indices $\bar l$ is called the λ-*admissible trigonometric hypergeometric space specialized at* z,λ and is denoted $\mathfrak{G}_{\mathrm{adm}}(z,\lambda)$.

For $\lambda \in \mathbb{C}^n$, a function $W(t, z, \mu) \in \mathfrak{G}$ is called λ-*admissible* if for all non-λ-admissible indices $\bar m \in \mathcal{Z}_l^n$, the functions $a_{\bar m}(\mu, u)$ in decomposition (7) are equal to zero.

For fixed $z, \lambda \in \mathbb{C}^n$, the space spanned over $\mathbb{C}$ by all λ-admissible holomorphic singular trigonometric weight functions is called the λ-*admissible singular hypergeometric trigonometric space specialized at* z, λ and is denoted $\mathfrak{G}^{\,sing}_{\mathrm{adm}}(z, \lambda)$.

3.5. **Hypergeometric integrals.** Fix $p \in \mathbb{C}$, $\operatorname{Re} p < 0$. Assume that the parameters $z, \lambda \in \mathbb{C}^n$ satisfy the condition $\operatorname{Re}(z_i + \lambda_i) < 0$ and $\operatorname{Re}(z_i - \lambda_i) > 0$ for all $i = 1, \dots, n$. For a rational weight function $w = w_{\bar l}(t, z, \lambda)$, $\bar l \in \mathcal{Z}_l^n$, and a singular trigonometric weight function $W(t, z, \lambda) \in \mathfrak{G}^{\,sing}$, define the *hypergeometric integral* $I(w, W)(z, \lambda)$ by the formula

$$I(w, W)(z, \lambda) = \int\limits_{\substack{\operatorname{Re} t_i = 0,\\ i=1,\dots,l}} \Phi(t, z, \lambda) w(t, z, \lambda) W(t, z, \lambda)\, d^l t, \tag{8}$$

where $d^l t = dt_1 \dots dt_l$.

The hypergeometric integral for generic z, λ and an arbitrary step p with negative real part is defined by analytic continuation with respect to z, λ and p, see [TV1], [MV1].

For a function $W(t, z, \lambda) \in \mathfrak{G}^{\,sing}$, let $\Psi_W(z, \lambda)$ be the following $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$-valued function

$$\Psi_W(z, \lambda) \;=\; \sum_{l_1 + \dots + l_n = l} I(w_{\bar l}, W)(z, \lambda)\, f^{l_1} v_1 \otimes \dots \otimes f^{l_n} v_n. \tag{9}$$

Theorem 2. *(Corollaries 5.25, 5.26 in* [TV1]*.) Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ *and* $\lambda_i \notin \Lambda^+$. *Then for generic values of* p *and for any function* $W \in \mathfrak{G}^{\,sing}$, *the function* $\Psi_W(z, \lambda)$ *is a meromorphic solution of the qKZ equation with values in* $(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})_l^{sing}$.

We always asume the following conditions on parameters $p \in \mathbb{C}$, $z, \lambda \in \mathbb{C}^n$:

$$\operatorname{Re} p < 0, \qquad 1 \notin p\mathbb{Z}, \tag{10}$$

$$\{s \,|\, s \in \mathbb{Z}_{>0}, s < 2\max\{\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_n\},\ s \le l\} \bigcap \{p\mathbb{Z}\} = \emptyset, \tag{11}$$

$$\{2\lambda_m - s \,|\, s \in \mathbb{Z}_{\ge 0},\ s < 2\operatorname{Re}\lambda_m,\ s < l\} \bigcap \{p\mathbb{Z}\} = \emptyset, \qquad m = 1, \dots, n, \tag{12}$$

$$z_k - z_m \pm (\lambda_k + \lambda_m) + s \notin \{p\mathbb{Z}\}, \qquad k, m = 1, \dots, n,\ k \ne m,\ s = 1 - l, \dots, l - 1. \tag{13}$$

For each $i \in \{1, \dots, n\}$ such that $\lambda_i \notin \Lambda^+$, assume

$$\{1, \dots, l\} \bigcap \{p\mathbb{Z}\} = \emptyset, \tag{14}$$

$$\{2\lambda_i - s \,|\, s = 0, 1, \dots, l - 1\} \bigcap \{p\mathbb{Z}\} = \emptyset. \tag{15}$$

Let $\lambda \in \mathbb{C}^n$. For a λ-admissible function $W \in \mathfrak{G}^{\,sing}$, consider a function

$$\Psi_W^{\mathrm{adm}}(z, \lambda) = \sum I(w_{\bar l}, W)(z, \lambda) f^{l_1} v_1 \otimes \dots \otimes f^{l_n} v_n, \tag{16}$$

where the sum is over all λ-admissible $\bar l \in \mathcal{Z}_l^n$.

Theorem 3. *(Corollaries 17, 18 in* [MV1]*.) Let $p \in \mathbb{C}$ and $\lambda \in \mathbb{C}^n$ satisfy conditions (10)-(12) and let $W \in \mathfrak{G}^{\,sing}$ be a λ-admissible function. Then*

1. The function $\Psi_W(z,\lambda)$ is a meromorphic solution of the qKZ equation with values in $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})_l^{sing}$.

2. The function $\Psi_W^{\mathrm{adm}}(z,\lambda)$ is a meromorphic solution of the qKZ equation with values in $(L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})_l^{sing}$.

The meromorphic solutions of the qKZ equation defined by (9) and (16) are called the *hypergeometric solutions.*

3.6. **Relations with representation theory.** Let $p \in \mathbb{C}$, $z, \lambda \in \mathbb{C}^n$ satisfy conditions (10)-(15).

The weight space $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})_l^*$ is identified with the hypergeometric rational space $\mathfrak{F}\,(z,\lambda)$ by the map

$$\mathfrak{a}(z,\lambda):\ (f^{l_1}v_1 \otimes \ldots \otimes f^{l_n}v_n)^* \mapsto w_{\bar{l}}(t,z,\lambda).$$

Here $\{(f^{l_1}v_1 \otimes \ldots \otimes f^{l_n}v_n)^* \,|\, l_1,\ldots,l_n \in \mathbb{Z}_{\geq 0}\}$ is the basis of $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^*$, dual to the standard basis of $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ given by $\{f^{l_1}v_1 \otimes \ldots \otimes f^{l_n}v_n \,|\, l_1,\ldots,l_n \in \mathbb{Z}_{\geq 0}\}$, see Lemma 4.5, Corallary 4.8 in [TV1].

For $\bar{l} \in \mathcal{Z}_l^n$, define the *weight coefficient* $c_{\bar{l}}(\lambda)$ by

$$c_{\bar{l}}(\lambda) = \prod_{m=1}^{n} \prod_{s=0}^{l_m-1} \frac{\sin(\pi(s+1)/p)\sin(\pi(2\lambda_m - s)/p)}{\sin(\pi/p)}.$$

Let $q = e^{\pi i/p}$. The weight space $(V_{\lambda_1}^q \otimes \ldots \otimes V_{\lambda_n}^q)_l$ is identified with the trigonometric hypergeometric space $\mathfrak{G}\,(z,\lambda)$ by the map

$$\mathfrak{b}(z,\lambda):\ f_q^{l_1}v_1^q \otimes \ldots \otimes f_q^{l_n}v_n^q \mapsto c_{\bar{l}}(\lambda)W_{\bar{l}}(t,z,\lambda), \tag{17}$$

see Lemma 4.17 and Corallary 4.20 in [TV1]. The subspace of singular vectors $(V_{\lambda_1}^q \otimes \ldots \otimes V_{\lambda_n}^q)_l^{sing}$ is identified with the singular hypergeometric space,

$$\mathfrak{b}(z,\lambda)((V_{\lambda_1}^q \otimes \ldots \otimes V_{\lambda_n}^q)_l^{sing}) = \mathfrak{G}^{\,sing}(z,\lambda),$$

see Corallary 4.21 in [TV1]. Moreover, the map $\mathfrak{b}(z,\lambda)$ is factorized to an isomorphism

$$\mathfrak{b}(z,\lambda)\ :\ (L_{\lambda_1}^q \otimes \ldots \otimes L_{\lambda_n}^q)_l \to \mathfrak{G}_{\,\mathrm{adm}}(z,\lambda),$$

given by the same formula (17). The subspace of singular vectors is identified with the λ-admissible singular hypergeometric trigonometric space,

$$\mathfrak{b}(z,\lambda)((L_{\lambda_1}^q \otimes \ldots \otimes L_{\lambda_n}^q)_l^{sing}) = \mathfrak{G}_{\,\mathrm{adm}}^{\,sing}(z,\lambda).$$

We have the hypergeometric map,

$$s(z,\lambda):\ (V_{\lambda_1}^q \otimes \ldots \otimes V_{\lambda_n}^q)_l^{sing} \to (V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})_l^{sing},$$

defined for $v^q \in (V_{\lambda_1}^q \otimes \ldots \otimes V_{\lambda_n}^q)_l^{sing}$ by $v^q \mapsto \Psi_{\mathfrak{b}(z,\lambda)v^q}(z,\lambda)$, cf. Theorem 14 in [MV1], [TV1].

This map is factorized to the hypergeometric map,

$$s(z,\lambda):\ (L_{\lambda_1}^q \otimes \ldots \otimes L_{\lambda_n}^q)_l^{sing} \to (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})_l^{sing},$$

defined for $v^q \in (L_{\lambda_1}^q \otimes \ldots \otimes L_{\lambda_n}^q)_l^{sing}$ by $v^q \mapsto \Psi_{\mathfrak{b}(z,\lambda)v^q}^{\mathrm{adm}}(z,\lambda)$, cf. [MV1].

Theorem 4. *(Theorem 24 in* [MV1]*.) Let p, z, λ satisfy conditions (10)-(15). Let also $\sum_{m=1}^{n} 2\lambda_m - 2l + k + 1 \notin p\mathbb{Z}_{<0}$ for $k = 1, \ldots, l-1$. Then the map $s(z,\lambda) : (L^q_{\lambda_1} \otimes \ldots \otimes L^q_{\lambda_n})^{sing}_l \to (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l$ is an isomorphism of vector spaces.*

In this paper we describe the image and kernel of the hypergeometric map $s(z,\lambda)$ when the resonance condition, $\sum_{m=1}^{n} 2\Lambda_m - 2l + p + k + 1 = 0$, holds for some $k \in \{1, \ldots, l-1\}$.

4. The hypergeometric solutions in the case of resonance

4.1. **The space of quantized conformal blocks.** Let L_{λ_i} be the irreducible $sl(2)$ module with highest weight $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$. Let

$$e(z) = T^{(2)}_{21} - T^{(1)}_{11} T^{(1)}_{21} \in Y(gl(2)). \tag{18}$$

Then, for each λ-admissible $\bar{l} \in \mathbb{Z}^n_{\geq 0}$, we have

$$e(z) f^{l_1} v_1 \otimes \ldots \otimes f^{l_n} v_n = \sum_{j=1}^{n} \left(z_j + h^{(j)} + \sum_{s=j+1}^{n} 2h^{(s)} \right) e^{(j)} f^{l_1} v_1 \otimes \ldots \otimes f^{l_n} v_n = \tag{19}$$

$$\sum_{j=1}^{n} (2\lambda_i - m_i + 1) l_i (z_j + \lambda_j - l_j + \sum_{s=j+1}^{n} 2(\lambda_s - l_s)) f^{l_1} v_1 \otimes \ldots \otimes f^{l_j - 1} v_j \otimes \ldots \otimes f^{l_n} v_n,$$

where $h^{(s)}, e^{(s)}$ are the elements $h, e \in sl(2)$ acting in the s-th factor.

Define the *space of quantized conformal blocks* $C(z)$. Let

$$C(z) = \{ m \in (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l \mid (e(z))^k m = 0 \}, \tag{20}$$

if the resonance condition,

$$\sum_{i=1}^{n} 2\lambda_i - 2l + p + k + 1 = 0, \tag{21}$$

holds for some $k \in \mathbb{N}$, and $C(z) = (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l$ otherwise.

Note that if $k > l$ then $C(z) = (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l$.

The subspace $(L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l$ is called a *resonance subspace* if the resonance condition (21) holds for some $k \in \mathbb{N}$, $k \leq l$.

For $x \in \mathbb{C}$, let

$$e_x(z) = e(z) + x T^{(1)}_{21} \in Y(gl(2)). \tag{22}$$

If (21) holds for some $k \in \mathbb{N}$, then the space $\{ m \in (L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n})^{sing}_l \mid (e_x(z))^k m = 0 \}$ does not depend on x and coincides with the space of conformal blocks $C(z)$, see [MV2].

Theorem 5. *(Theorem 2 in* [MV2]*.) The space of conformal blocks $C(z)$ is invariant with respect to the qKZ connection,*

$$K_i(z) C(z) = C(z_1, \ldots, z_i + p, \ldots, z_n),$$

as well as with respect to permutations of variables,

$$PR_{M_i M_{i+1}}(z_i - z_{i+1}) C(z) = C(z_1, \ldots, z_{i+1}, z_i, \ldots, z_n).$$

4.2. **The image of $s(z,\lambda)$.**

Theorem 6. *Let p, z, λ satisfy conditions (10)-(15) and let L_{λ_i} be the irreducible $sl(2)$ module with highest weight λ_i, $i = 1, \dots, n$. Then the image of the map $s(z,\lambda)$ belongs to the space of conformal blocks $C(z)$, i.e. every hypergeometric solution (16) takes values in the space of conformal blocks.*

Example. Let $l = k = 1$, $n = 3$. Then the solutions of the qKZ equation have the form

$$\Psi^{\pm}(z,\lambda) = I_1^{\pm}(z,\lambda)\, f v_1 \otimes v_2 \otimes v_3 + I_2^{\pm}(z,\lambda)\, v_1 \otimes f v_2 \otimes v_3 + I_3^{\pm}(z,\lambda)\, v_1 \otimes v_2 \otimes f v_3,$$

where

$$I_m^{\pm}(z,\lambda) = \int \Phi(t,z,\lambda) w_m(t,z,\lambda) \left(\prod_{j=1}^{3} \sin(\pi(t - z_j - \lambda_j)/p) \right)^{-1} e^{\pm \pi i t/p} dt,$$

$m = 1, 2, 3$, and the weight functions $w_m(t,z,\lambda)$ are given by (6). According to Theorem 6, if $p + 2\sum_{j=1}^{3} \lambda_j = 0$ then the coordinate functions $I_m^{\pm}(z,\lambda)$ satisfy the algebraic equation,

$$\sum_{i=1}^{3} 2\lambda_i \left(z_i + \lambda_i + 2 \sum_{j=i+1}^{3} \lambda_j \right) I_i^{\pm}(z,\lambda) = 0.$$

Proof: Theorem 6 is trivial for non-resonance subspaces $(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})_l^{sing}$. Let $(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})_l^{sing}$ be a resonance subspace, so that $\sum_{i=1}^{n} 2\lambda_i - 2l + p + k + 1 = 0$ holds for some $k \in \mathbb{N}$, $k \le l$. Let $W \in \mathfrak{G}^{\,sing}$ be an admissible trigonometric weight function. We have to prove that $\Psi_W^{\rm adm}(z,\lambda) \in C(z)$, where $\Psi_W^{\rm adm}(z,\lambda)$ is given by formula (9). Let $e_p(z) \in Y(gl(2))$ be given by (22). We have to prove that $(e_p(z))^k \Psi_W^{\rm adm}(z,\lambda) = 0$, i.e all coordinates of this vector equal zero. It is equivalent to the statement that for each λ-admissible $\bar m \in \mathcal{Z}_{l-k}^n$, we have

$$I\left(\mathfrak{a}(z,\lambda)(f_p(z))^k (f^{m_1} v_1 \otimes \dots \otimes f^{m_n} v_n)^*,\, W \right) = 0,$$

where $f_p(z) \in \mathrm{End}((L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^*)$ is defined for λ-admissible $\bar m \in Z_{\ge 0}^n$, by

$$f_p(z)(f^{m_1} v_1 \otimes \dots \otimes f^{m_n} v_n)^* = \sum_{i=1}^{n} \Bigg((m_i + 1)(2\lambda_i - m_i) \times$$

$$\times (z_i + p + \lambda_i - m_i + \sum_{j=i+1}^{n} 2(\lambda_j - m_j)) \Bigg) (f^{m_1} v_1 \otimes \dots \otimes f^{m_i+1} v_i \otimes \dots \otimes f^{m_n} v_n)^*,$$

cf. formula (19).

For a function $\varphi(t) = \varphi(t_1, \dots, t_l)$, define the *discrete partial derivatives* $(D_i\varphi)(t)$, $i = 1, \dots, l$, by

$$(D_i\varphi)(t) = \varphi(t_1, \dots, t_i + p, \dots, t_l) - \varphi(t).$$

Recall that $w_{\bar l} = \sum_{\sigma \in \mathbb{S}^l} [\eta_{\bar l}]_\sigma^{rat}$, where $\eta_{\bar l}$ is defined in (5).

For fixed $z, \lambda \in \mathbb{C}^n$ and $p \in \mathbb{C}$, introduce a linear operator $\tilde{f}_p(z)$ acting in the space spanned over $\mathbb{C}$ by all functions $\eta_{\bar{l}}(t, z, \lambda)$, $\bar{l} \in \mathbb{Z}^n_{\geq 0}$, by the formula

$$\tilde{f}_p(z)\eta_{\bar{m}} = \sum_{i=1}^{n}(m_i+1)(2\lambda_i - m_i)(z_i + p + \lambda_i - m_i + \sum_{j=i+1}^{n} 2(\lambda_j - m_j))\eta_{(m_1,\dots,m_i+1,\dots,m_n)}.$$

Lemma 7. *Let $\bar{m} \in \mathcal{Z}^n_{l-k}$, then*

$$\sum_{\sigma\in\mathbb{S}}^{l}\left[\sum_{i=0}^{k}\left(\binom{k}{i}\left(\prod_{j=1}^{k-i}(p+\sum_{i=1}^{n}2(\lambda_i-l)+k+j)\right)\right)((\tilde{f}_p(z))^i\eta_{\bar{m}})(t_{k+1-i},\dots,t_l)\right]_{\sigma}^{rat} =$$

$$= \sum_{\sigma\in\mathbb{S}}^{l}[(\Phi(t,z,\lambda))^{-1}\mu_{\bar{m}}(t,z,\lambda)]^{rat}_{\sigma}, \tag{23}$$

where

$$\mu_{\bar{m}}(t,z,\lambda) = (D_1(\varphi_1(D_2(\varphi_2\dots(D_k(\varphi_k\eta_{\bar{m}}(t_{k+1},\dots,t_l,z,\lambda)))\dots)))),$$

$$\varphi_a(t,z,\lambda) = t_a\prod_{i=1}^{n}\frac{\Gamma((t_a-z_i+\lambda_i)/p)}{\Gamma((t_a-z_i-\lambda_i)/p)}\prod_{j=a+1}^{l}\frac{\Gamma((t_a-t_j-1)/p)}{\Gamma((t_a-t_j+1)/p)}.$$

Example. Let $l = k = 1$. Then

$$D_1(t\Phi) =$$

$$\left(p+\sum_{i=1}^{n}2\lambda_i\right)\Phi(t,z,\lambda)+\sum_{i=1}^{n}2\lambda_i\left(z_i+p+\lambda_i+\sum_{j=i+1}^{n}2\lambda_j\right)w_i(t,z,\lambda)\Phi(t,z,\lambda),$$

where $w_i(t,z,\lambda)$ are the weight functions given by (6).
Proof: Introduce the numbers $y_1,\dots,y_{n+l-1}$ by

$$(y_1,\dots,y_{n+l-1}) = (t_2,\dots,t_{l_1+1},z_1,t_{l_1+2},\dots,t_{l_1+l_2+1},z_2,t_{l_1+l_2+2},\dots,z_n),$$

and the numbers $\Delta_1,\dots,\Delta_{n+l-1}$ by

$$(\Delta_1,\dots,\Delta_{n+l-1}) = (-2,\dots,-2,2\lambda_1,-2,\dots,-2,2\lambda_2,-2,\dots,2\lambda_n).$$

Order the numbers $y_1,\dots,y_{n+l-1}$ by the rule: $y_i \prec y_j$ if and only if $i<j$.

For $i = 1,\dots,n+l-1$, introduce

$$\eta_i(t_1,y,\Delta) = \frac{1}{t_1-y_i-\Delta_i}\prod_{j=1}^{i-1}\frac{t_1-y_j+\Delta_j}{t_1-y_j-\Delta_j},$$

cf. formula (6).

For $i = 1,\dots,n+l-1$, we have the following equalities, proved by induction on i,

$$\prod_{s=1}^{i}\frac{t_1-y_s+\Delta_s}{t_1-y_s-\Delta_s}-1 = \sum_{s=1}^{i}2\Delta_s\eta_s(t_1,y,\Delta),$$

$$t_1\eta_i(t_1,y,\Delta)-1 = \sum_{j=1}^{i-1}2\Delta_j\eta_j(t_1,y,\Delta)+(y_i+\Delta_i)\eta_i(t_1,y,\Delta).$$

For $\bar{l} \in \mathcal{Z}^n_{l-1}$, $j = 1, \dots, n+l-1$, we have

$$\sum_{\sigma \in \mathbb{S}^l} [\eta_{\bar{l}}(t_2, \dots, t_l, z, \lambda)\eta_j(t_1, y, \Delta)]^{rat}_\sigma = \sum_{\sigma \in \mathbb{S}^l} [a_j \eta_{(l_1, \dots, l_i+1, \dots, l_n)}(t, z, \lambda)]^{rat}_\sigma,$$

where $i \in \{1, \dots, n\}$ is such that $z_{i-1} \prec y_j \preceq z_i$; $a_j = l_i + 1$ if $y_j = z_j$ and $a_j = \frac{1}{2}(l_i + 1)$ otherwise.

Let $j \in \{1, \dots, n+l-1\}$ be such that $z_{i-1} \prec y_j = t_s \prec z_i$. Then for $\bar{l} \in \mathcal{Z}^n_{l-1}$, $j = 1, \dots, n+l-1$, we have

$$\sum_{\sigma \in \mathbb{S}^l} [\eta_{\bar{l}}(t_2, \dots, t_l, z, \lambda) y_j \eta_j(t_1, y, \Delta)]^{rat}_\sigma = \sum_{\sigma \in \mathbb{S}^l} [(z_i + \lambda_i)\eta_{\bar{l}}(t, z, \lambda)\eta_j(t_1, y, \Delta)]^{rat}_\sigma.$$

Using the above formulas, we compute

$$\sum_{\sigma \in \mathbb{S}^l} \left[(t_1 + p)\eta_{\bar{l}}(t_2, \dots, t_l, z, \lambda) \left(\prod_{s=1}^{n+l-1} \frac{t_1 - y_s + \Delta_s}{t_1 - y_s - \Delta_s} - 1 \right) \right]^{rat}_\sigma = \sum_{i=1}^n 2\Lambda_i - 2(l-1) +$$

$$+ \sum_{i=1}^n (l_i + 1)(2\lambda_i - l_i) \left(z_i + \lambda_i - l_i + \sum_{j=i+1}^n 2(\lambda_i - l_i) \right) \sum_{\sigma \in \mathbb{S}^l} [\eta_{(l_1, \dots, l_i+1, \dots, l_n)}(t, z, \lambda)]^{rat}_\sigma.$$

This proves the Lemma when $k = 1$. Lemma 7 for other k is proved similarly by induction on k. □

We deduce the Theorem from Lemma 7. Under resonance condition (21), the terms in the left hand side of (23) corresponding to $i = 1, \dots, k-1$ equal zero, and the only non-zero term (corresponding to $i = k$) equals $\mathfrak{a}(z, \lambda)(f(z))^k (f^{m_1} v_1 \otimes \dots \otimes f^{m_n} v_n)^*$. So,

$$I\left(\mathfrak{a}(z, \lambda)(f(z))^k (f^{m_1} v_1 \otimes \dots \otimes f^{m_n} v_n)^*, W\right)(z, \lambda) =$$

$$\int \Phi(t, z, \lambda) \left(\sum_{\sigma \in \mathbb{S}}^{l} [(\Phi(t, z, \lambda))^{-1} \mu_{\bar{m}}(t, z, \lambda)]^{rat}_\sigma \right) W d^l t =$$

$$l! \int \Phi(t, z, \lambda)(\Phi(t, z, \lambda))^{-1} \mu_{\bar{m}}(t, z, \lambda)) W d^l t = l! \int \mu_{\bar{m}}(t, z, \lambda)) W d^l t,$$

where the integration is over a suitable contour, see [MV1], and in the second equality we use the symmetry of the initial integrand, see the Remark in Section 3.2.

The function $W(t, z, \lambda)$ is p-periodic with respect to each of $t_1, \dots, t_l$. According to Lemma 7, $\mu_{\bar{m}}(t, z, \lambda) = D_1 g_{\bar{m}}(t, z, \lambda)$ for some function $g_{\bar{m}}(t, z, \lambda)$. Hence $\mu_{\bar{m}}(t, z, \lambda) W(t, z, \lambda) = D_1(g_{\bar{m}} W)(t, z, \lambda)$. Note that the function $g_{\bar{m}}(t, z, \lambda)$ has a polynomial growth as t_j goes to $\pm i\infty$. All functions $W(t, z, \lambda) \in \mathfrak{G}^{sing}(z, \lambda)$ decay exponentially as t_j goes to $\pm i\infty$. Therefore, all the integrals converge.

If $\lambda_i \ll 0$ for all $i = 1, \dots, n$, then the integral $\int\limits_{\substack{\mathrm{Re}\, t_i = 0, \\ i=1,\dots,l}} D_1(g_{\bar{m}} W)(t, z, \lambda) d^l t$ is zero, since the function $g_{\bar{m}}(t, z, \lambda) W(t, z, \lambda)$ has no poles between $\{t \in \mathbb{C}^l \,|\, \mathrm{Re}\, t_i = 0,\ i = 1, \dots, l\}$ and $\{t \in \mathbb{C}^l \,|\, \mathrm{Re}\, t_1 = \mathrm{Re}\, p,\ \mathrm{Re}\, t_i = 0,\ i = 2, \dots, l\}$. The integral is a meromorphic function of λ, hence it is zero for all λ, cf. Lemma 9.5 in [TV1]. □

4.3. **The kernel of** $s(z,\lambda)$**.** Let p,z,λ satisfy conditions (10)-(15). Let L_{λ_i} be the irreducible $sl(2)$ module with highest weight $\lambda_i\in\mathbb{C}$ and let $L^q_{\lambda_i}$ be the corresponding $U_q sl(2)$ module with highest weight q^{λ_i}, $i=1,\dots,n$.

Let $(L_{\lambda_1}\otimes\ldots\otimes L_{\lambda_n})^{sing}_l$ be a resonance subspace, so that $\sum\limits_{i=1}^n 2\lambda_i-2l+p+k+1=0$ holds for some $k\in\mathbb{N}$, $k\le l$.

Lemma 8. *Let* $v^q\in(L^q_{\lambda_1}\otimes\ldots\otimes L^q_{\lambda_n})^{sing}_{l-k}$. *Then* $(f_q)^k v^q\in(L^q_{\lambda_1}\otimes\ldots\otimes L^q_{\lambda_n})^{sing}_{l}$.

Proof: We have

$$e_q(f_q)^k v^q=[\sum_{i=1}^n 2\lambda_i-2(l-k)-k+1]_q[k]_q(f_q)^{k-1}v^q,$$

where $[k]_q=(q^k-q^{-k})/(q-q^{-1})$. Recall that $q=e^{\pi i/p}$, so $[\sum\limits_{i=1}^n 2\lambda_i-2(l-k)-k+1]_q=[-p]_q=0$. $\square$

Theorem 9. *Let* $v^q\in(L^q_{\lambda_1}\otimes\ldots\otimes L^q_{\lambda_n})^{sing}_{l}$ *have the form* $v^q=(f_q)^k\tilde v^q$ *for some* $\tilde v^q\in(L^q_{\lambda_1}\otimes\ldots\otimes L^q_{\lambda_n})^{sing}_{l-k}$. *Then the hypergeometric solution* $s(z,\lambda)\,v^q$ *equals zero.*

Proof: Let $W=\mathfrak{b}(z,\lambda)v^q$. We need to prove that $\Psi^{\mathrm{adm}}_W(z)$ given by (16) is zero.

The space $\bigoplus\limits_{l=0}^{\infty}\mathfrak{G}^{\,n}_{\,l}(z,\lambda)$ has a $U_q sl_2$ module structure such that the map $\mathfrak{b}(z,\lambda)$ is an intertwiner of $U_q sl(2)$ modules, see [TV1]. The action of $U_q sl(2)$ in the space $\bigoplus\limits_{l=0}^{\infty}\mathfrak{G}^{\,n}_{\,l}(z,\lambda)$ is given by formulas (4.16) in [TV1]. In particular, for any $X(t,z,\lambda)\in\mathfrak{G}^{\,n}_{\,l}(z,\lambda)$, we have

$$(f_qX)(t_1,\dots,t_{l+1})=\exp\left(-\pi i(l+\sum_{m=1}^n\lambda_m)/p\right)\sum_{a=1}^{l+1}\Bigg[X(t_2,\dots,t_{l+1})\Bigg(\exp(2\pi i l/p)\times$$

$$\times\prod_{m=1}^n\frac{\sin(\pi(t_1-z_m+\lambda_m)/p)}{\sin(\pi(t_1-z_m-\lambda_m)/p)}\prod_{b=2}^{l+1}\frac{\sin(\pi(t_1-t_b-1)/p)}{\sin(\pi(t_1-t_b+1)/p)}-\exp(2\pi i\sum_{m=1}^n\lambda_m/p)\Bigg)\Bigg]^{trig}_{(1,a)},$$

where $(1,a)\in\mathbb{S}^{l+1}$ are transpositions.

Therefore, $W=\mathfrak{b}(z,\lambda)(f_q)^k\tilde v^q$ has the form $W=\sum\limits_{\sigma\in\mathbb{S}^l}[\Upsilon]^{trig}_\sigma$, where

$$\Upsilon(t_1,\dots,t_l,z,\lambda)=a(\lambda)X(t_{k+1},\dots,t_l)\prod_{i=1}^k\Bigg(\exp\left(2\pi i(l-i)/p\right)\times \tag{24}$$

$$\times\prod_{m=1}^n\frac{\sin(\pi(t_i-z_m+\lambda_m)/p)}{\sin(\pi(t_i-z_m-\lambda_m)/p)}\prod_{b=i+1}^{l}\frac{\sin(\pi(t_i-t_b-1)/p)}{\sin(\pi(t_i-t_b+1)/p)}-\exp(2\pi i\sum_{m=1}^n\lambda_m/p)\Bigg),$$

for some $X\in\mathfrak{G}^{\,sing}_{\,l-k}(z,\lambda)$ and some holomorphic function $a(\lambda)$.

Fix $w=w_{\bar l}$, where $\bar l\in\mathcal{Z}^n_l$. Fix λ such that $\operatorname{Im}\lambda_i\ne0$, and $\sum\limits_{i=1}^n 2\lambda_i-2l+p+k+1=0$. We need to prove that $I(w,W)(z,\lambda)=0$, where $I(w,W)$ is the hypergeometric integral

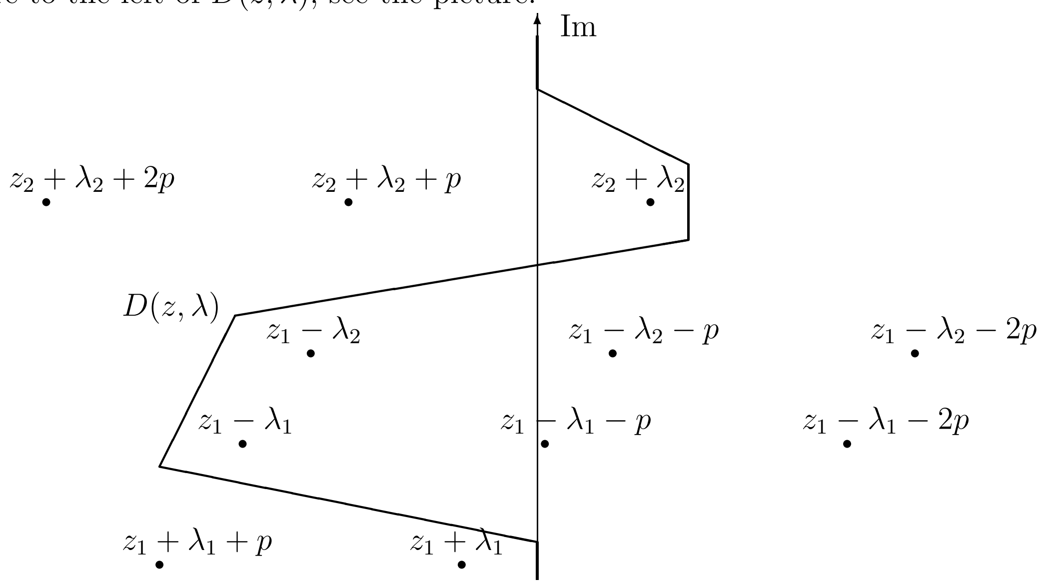
defined in Section 3.5. The integrand $\Phi w W$ is a meromorphic function of t, z, λ with simple poles located at most at

$$t_i = z_j \pm (\lambda_j + kp), \qquad k \in \mathbb{Z}_{\geq 0}, \qquad i = 1, \ldots, l, \qquad j = 1, \ldots, n,$$

and at

$$t_i - t_j = \pm(1 - kp), \qquad k \in \mathbb{Z}_{\geq 0}, \qquad i, j = 1, \ldots, l, \qquad i < j. \tag{25}$$

Let $z \in \mathbb{C}^n$ be such that $|\mathrm{Im}\,\lambda_i| \ll \mathrm{Im}\,z_1 \ll \ldots \ll \mathrm{Im}\,z_n$. Let p be a real negative number. Let $D(z, \lambda) \subset \mathbb{C}$ be a curve such that $D(z, \lambda)$ coincides with the imaginary line at a neighbourhood of infinity, $D(z, \lambda)$ consists of straight non-horisontal segments, the points $z_j - \lambda_j$, $j = 1, \ldots, n$, are to the right of $D(z, \lambda)$ and the points $z_j + \lambda_j$, $j = 1, \ldots, n$, are to the left of $D(z, \lambda)$, see the picture.



The analytic continuation of the hypergeometric integral $I(w, W)$ with respect to λ gives

$$I(w, W) = \int\limits_{\substack{t_i \in D(z,\lambda) \\ i=1,\ldots,l}} \Phi w W d^l t,$$

cf. [MV1].

Due to the symmetry of the function $\Phi w W$, it is enough to prove that the integral of $\Phi w \Upsilon$ equals zero, $I(w, \Upsilon) = 0$, see the Remark in Section 3.2.

Let A, B be large positive real numbers. Let $D_A(z, \lambda)$ be the curve $D(z, \lambda)$ truncated at infinity, namely, $D_A(z, \lambda) = \{x \in D(z, \lambda) \,|\, |\mathrm{Im}\,x| \leq A\}$. Let

$$I_{AB}(w, \Upsilon) = \int\limits_{\substack{t_j \in D_B(z,\lambda) \\ j=k+1,\ldots,l}} \int\limits_{\substack{t_i \in D_A(z,\lambda) \\ i=1,\ldots,k}} \Phi w \Upsilon d^l t.$$

We have $I_{AB}(w, \Upsilon) \to I(w, \Upsilon)$ when $A, B \to +\infty$.

Let $D_A^+ = \{x \in \mathbb{C} \,|\, \mathrm{Re}\,x \geq 0,\, |x| = A\}$ and $D_A^- = \{x \in \mathbb{C} \,|\, \mathrm{Re}\,x \leq 0\, |x| = A\}$ are halves of the circle of radius A centered at the origin.

The function $\Upsilon(t, z, \lambda)$ is given by (24) as a product of k factors, and each of the factors is a sum of two terms. Represent the function $\Upsilon(t, z, \lambda)$ as a sum of 2^k terms,

$\Upsilon = \sum_{s=1}^{2^k} \Upsilon_{(s)}$. Let $E_{(s)} \subset \{1,\dots,k\}$ consist of the indices $i \in \{1,\dots,k\}$ such that $\Upsilon_{(s)}$ contains the factor

$$\prod_{m=1}^{n} \frac{\sin(\pi(t_i - z_m + \lambda_m)/p)}{\sin(\pi(t_i - z_m - \lambda_m)/p)}.$$

The function $\Phi w \Upsilon_{(s)}$ has no pole at

$$t_i = z_j - (\lambda_j + kp), \qquad k \in \mathbb{Z}_{\geq 0}, \qquad i \in E_{(s)}, \qquad j = 1,\dots,n,$$

$$t_i = z_j + (\lambda_j + kp), \qquad k \in \mathbb{Z}_{\geq 0}, \qquad i \in \{1,\dots,k\},\, i \notin E_{(s)}, \qquad j = 1,\dots,n.$$

Consider the integral $I_{AB}(w, \Upsilon_{(s)})$. We move the contours of integration with respect to t_i, $i = 1,\dots,k$, to the right if $i \notin E_{(s)}$ and to the left if $i \in E_{(s)}$. It easy to see that we do not encounter poles (25), therefore, we get

$$I_{AB}(w, \Upsilon_{(s)}) = \int\limits_{\substack{t_a \in D_B(z,\lambda)\\ a=k+1,\dots,l}} \int\limits_{\substack{t_i \in D_A(z,\lambda)\\ i=1,\dots,k}} \Phi w \Upsilon_{(s)} d^l t = \int\limits_{\substack{t_a \in D_B(z,\lambda)\\ a=k+1,\dots,l}} \int\limits_{\substack{t_j \in D_A^+(z,\lambda)\\ j \notin E_{(s)}}} \int\limits_{\substack{t_i \in D_A^-(z,\lambda)\\ i \in E_{(s)}}} \Phi w \Upsilon_{(s)} d^l t.$$

Now, we fix $t_{k+1},\dots,t_l$ and estimate the integrand using the Stirling formula as $A \to +\infty$. We have

$$\Phi(t,z,\lambda)\Upsilon_{(s)}(t,z,\lambda) \sim c_1 A^{k \sum\limits_{i=1}^{n} 2\lambda_i/p - 2(l-k)k/p - k(k-1)/p}, \qquad w(t,z,\lambda) \sim c_2 A^{-k},$$

where $c_1, c_2 \in \mathbb{C}$ are some quantities independent on A. Multiplying and using the resonance condition, we get

$$\Phi w \Upsilon_{(s)} \sim c_1 c_2 A^{k/p(\sum_{i=1}^{n} 2\lambda_i - 2l + k + 1) - k} = c_1 c_2 A^{-2k}.$$

We integrate with respect to each of $t_1,\dots,t_k$ over a semi-circle of radius A, hence the integral with respect to $t_1,\dots,t_k$ can be estimated by

$$\int\limits_{\substack{t_j \in D_A^+(z,\lambda)\\ j \notin E_{(s)}}} \int\limits_{\substack{t_i \in D_A^-(z,\lambda)\\ i \in E_{(s)}}} \Phi w \Upsilon_{(s)} d^l t \sim c_1 c_2 A^{-2k} (\pi A)^k = c_3 A^{-k},$$

for some $c_3 \in \mathbb{C}$ independent on A. Therefore, $I_{AB}(w, \Upsilon) \to 0$ as $A \to +\infty$. Thus, $I(w,\Upsilon)(z,\lambda) = I(w,W)(z,\lambda) = 0$ for our choice of z, λ.

The integral $I(w,W)$ is a meromorphic function of z, λ, therefore, $I(w,W)(z,\lambda) = 0$ for all $z, \lambda \in \mathbb{C}^n$.

This proof can be easily adjusted to any $p \in \mathbb{C}$ with properties (10),(14). □

4.4. **Finite dimensional representations and an integer step.** Let $\lambda_i \in \Lambda^+$, $i = 1,\dots,n$, are non-negative half-integers. Let $-p, l \in \mathbb{Z}_{>0}$, and

$$2\lambda_i \leq -p - 2, \qquad i = 1,\dots,n. \tag{26}$$

Note that conditions (10)-(12) and (14)-(15) are automatically satisfied.

Let

$$l \leq 2\lambda_i, \qquad i = 1,\dots,n. \tag{27}$$

Theorem 10. *Let p, z, λ satisfy conditions (10)-(15), (26) and (27). Let the resonance condition (21) hold for some $k \in \mathbb{N}$. Let L_{λ_i} be the finite dimesional irreducible module with highest weight λ_i and let $L^q_{\lambda_i}$ be the corresponding $U_q sl(2)$ module with highest weight q^{λ_i}, $i = 1, \dots, n$. Then for $v^q \in (L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_l$, the hypergeometric solution $s(z,\lambda)v^q$ equals zero if and only if v^q has the form $v^q = (f_q)^k \tilde{v}^q$ for some $\tilde{v}^q \in (L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}$.*

Proof: The determinant of the hypergeometric pairing is given by formula (5.15) in [TV1]. It follows from formula (5.15) in [TV1] that under conditions of the Theorem the dimension of the kernel of the map $s(z,\lambda)$ is not greater than

$$\binom{n+l-k-2}{n-2} = \dim(V^q_{\lambda_1} \otimes \dots \otimes V^q_{\lambda_n})^{sing}_{l-k} = \dim(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}.$$

Under conditions of Theorem 10 the operator $(f_q)^k$ acting on $(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}$ is non-degenerate, so, Theorem 10 follows from Theorem 9 and Lemma 8. $\square$

Corollary 11. *Under assumptions of Theorem 10, the space of hypergeometric solutions is naturally identified with the space*

$$(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_l / (f_q)^k ((L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}).$$

Corollary 11 follows from Theorem 10.

Theorem 12. *Let p, z, λ satisfy conditions (10)-(15), (26) and (27). Let the resonance condition (21) hold for some $k \in \mathbb{N}$. Let L_{λ_i} be the finite dimesional irreducible module with highest weight λ_i and let $L^q_{\lambda_i}$ be the corresponding $U_q sl(2)$ module with highest weight q^{λ}, $i = 1, \dots, n$. Then the image of the map $s(z,\lambda) : (L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_l \to (L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^{sing}_l$ coincides with the space of conformal blocks $C(z)$.*

Proof: Recall that for the differential KZ connection the space of conformal blocks $N(z)$ is defined as the kernel of the operator $(E(z))^k$ acting in a resonance subspace $(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^{sing}_l$, where

$$E(z) f^{l_1} v_1 \otimes \dots f^{l_n} v_n = \sum_{i=1}^{n} l_i (2\lambda_i - l_i + 1) z_i \, f^{l_1} v_1 \otimes \dots \otimes f^{l_i - 1} v_i \otimes \dots \otimes f^{l_n} v_n,$$

see [FSV1-3]. By Theorem 3.4 in [Fn],

$$\dim N(z) = \dim(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_l - \dim(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}.$$

Lemma 13. *The dimensions of the spaces of conformal blocks in the differential case and difference case are related by* $\dim N(z) \geq \dim C(z)$.

Proof: For a function $\psi(z)$ with values in $(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})_{l-k}$, denote $\psi_{\bar{l}}(z)$ its $\bar{l}$-th coordinate function with respect to the basis $f^{l_1} v_1 \otimes \dots f^{l_n} v_n$, $\bar{l} \in \mathcal{Z}^n_{l-k}$.

Let $r = \dim(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^{sing}_l - \dim N(z)$ be the dimension of the image of the operator $(E(z))^k$ acting in $(L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^{sing}_l$. Then there exist vectors $w_1, \dots, w_r \in (L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})^{sing}_l$ independent on z such that for generic $z \in \mathbb{C}^n$, their images $(E(z))^k w_1, \dots, (E(z))^k w_r \in (L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n})_{l-k}$ are linearly independent. It means that

there are indices $\bar{l}_1, \dots, \bar{l}_r \in \mathcal{Z}^n_{l-k}$ such that the matrix $F(z) = \{(E(z))^k w_i)_{\bar{l}_j}\}^r_{j=1}$ is non-degenerate. Note that the determinant of $F(z)$ is a non-zero homogeneous polynomial of $z_1, \dots, z_n$ of degree kr.

Consider the matrix $G = \{(e(z))^k w_i)_{\bar{l}_j}\}^r_{j=1}$. We have

$$\det G(z) = \det F(z) + g(z),$$

where $g(z)$ is a polynomial in $z_1, \dots, z_n$ of degree less than kr. Therefore, $\det G(z) \neq 0$. □

By Theorem 6, the image of the map $s(z, \lambda)$ belongs to the subspace of conformal blocks $C(z)$. By Theorem 10, the dimension of the image of the map $s(z, \lambda)$ is equal to the dimension of $N(z)$, which not smaller than the dimension of $C(z)$ by Lemma 13. Hence, $\dim C(z) = N(z)$ and the map $s(z, \lambda)$ is onto the subspace of conformal blocks $C(z)$. □

We expect that Theorems 10 and 12 hold without the assumption $l \leq 2\lambda_i$, $i = 1, \dots, n$.

Corollary 14. *Under assumptions of Theorem 12, the dimensions of the subspaces of conformal blocks in the differential case and difference case are the same* $\dim N(z) = \dim C(z)$ *and equal* $\dim(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_l - \dim(L^q_{\lambda_1} \otimes \dots \otimes L^q_{\lambda_n})^{sing}_{l-k}$.

Corollary 14 follows from the proof of Theorem 12.

Corollary 15. *Under assumptions of Theorem 12, the map $s(z, \lambda)$ is an isomorphism of vector spaces:*

$$s(z, \lambda) \,:\, (L^q_1 \otimes \dots \otimes L^q_n)^{sing}_l / f^k_q (L^q_1 \otimes \dots \otimes L^q_n)^{sing}_{l-k} \,\to\, C(z).$$

Corollary final2 follows from Theorems 10 and 12.

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